Beta random variable

$$X \sim \mathrm{Beta}(\alpha, \beta): \quad f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

if $0 < x < 1$ (else $= 0$)

$$\therefore \boxed{\alpha = \beta = 1 \longrightarrow X \sim U[0,1].}$$

(n-dim, Dirichlet pdf)

Defn: Beta function $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$ $\alpha > 0$, $\beta = 0$

$$= \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(see below)

★ **Thrm:** $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du$ $\quad \alpha>0,\ \beta>0$

$$= \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Prf: $\Gamma(\alpha)\cdot\Gamma(\beta) = \int_{x=0}^{x=\infty} e^{-x} x^{\alpha-1}\,dx \int_{y=0}^{y=\infty} e^{-y} y^{\beta-1}\,dy$

$$\overset{\text{Fubini}}{=} \int_{y=0}^{y=\infty}\left[\int_{x=0}^{x=\infty} e^{-(x+y)} x^{\alpha-1} y^{\beta-1}\,dx\right] dy$$

$$= \int_{y=0}^{y=\infty}\int_{x=0}^{x=\infty} f(x(u,v),\ y(u,v))\,dx\,dy$$

Let: $x = x(u,v) = uv \qquad y = y(u,v) = u(1-v)$ $\quad$ double substitution

Given: $0 < x < \infty$ and $0 < y < \infty$

$\therefore\ 0 < x+y = u\cdot v + u(1-v) = \cancel{uv} + u - \cancel{uv} = u$

$\therefore\ u > 0$

$\therefore\ x > 0 \longrightarrow uv > 0$

$\therefore\ v > 0$ since $u > 0$

$\therefore\ u < \infty$ since $x = u\cdot v < \infty$ and $v > 0$

$\therefore\ 0 < u < \infty \longleftarrow$ first limit of integration

$\therefore\ y > 0 \longrightarrow u(1-v) > 0 \qquad \therefore\ 1-v > 0$ (since $u>0$)

$\therefore\ v < 1 \qquad \therefore\ 0 < v < 1 \longleftarrow$ second limit of integration

$$\therefore \Gamma(\alpha)\cdot\Gamma(\beta) \overset{\text{CVT}}{=} \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

(change of variable theorem)

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\; u\cdot du\,dv.$$

since $\underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{absolute determinant of Jacobian matrix}} = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right| = \left|\begin{matrix} v & u \\ 1-v & -u\end{matrix}\right|$ since $x = u\cdot v$, $y = u - uv$

$$= \left| -\cancel{vu} - u + \cancel{vu}\right| = |-u| = u.$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u}(uv)^{\alpha-1}(u(1-v))^{\beta-1} u \; du\,dv$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u} u^{\alpha-1} v^{\alpha-1} u^{\beta-1}(1-v)^{\beta-1} u^{1}\; du\,dv$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u} u^{\alpha-1+\beta-1+1}\,du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1}\,dv\right]$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u} u^{(\alpha+\beta)-1}\,du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1}\,dv\right]$$

$$= \Gamma(\alpha+\beta)\cdot B(\alpha,\beta)$$

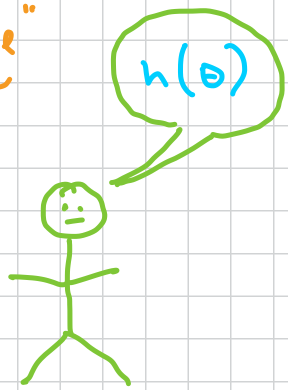
$$\therefore B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

QED

Bayesian Inference

Idea: Parameter θ becomes a r.v. $\Theta \sim h(\theta)$.

"Subjective"



"Objective"

experiment data.

likelihood: $g(x|\theta)$

$\therefore$ Find posterior pdf $f(\theta|x)$

Bayes theorem:

$$f(\theta|x) = \frac{h(\theta)\cdot g(x|\theta)}{\int_{-\infty}^{\infty} h(\theta)\cdot g(x|\theta)\, d\theta}$$

$$\propto h(\theta)\cdot g(x|\theta)$$

Conjugacy: $f(\theta|x)$ has same pdf form as $h(\theta)$

$\therefore$ use $f(\theta|x)$ as new prior and iterate

Thm: (Beta is conjugate to binomial)

If $h(\theta) = \text{Beta}(\alpha, \beta)$ - prior

$g(x|\theta) = \text{binomial}(n, x, \theta)$ - likelihood

(Bernoulli trials)

then $f(\theta|x) = \text{Beta}(\alpha + x, \beta + n - x)$ - posterior.

Ex: Coin flip Unknown success probability

$\theta = P(\text{heads})$

- assume beta prior $h(\theta) = \text{Beta}(\alpha, \beta)$
- flip coin n-times $\leftarrow$ experiment

observe x heads, $n-x$ tails.

- Conclude: $f(\theta|x) = \text{Beta}(\alpha + x, \beta + n - x)$

Prf:

$$\theta \sim h(\theta) = \text{Beta}(\alpha, \beta)$$

prior

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot \theta^{\alpha-1}(1-\theta)^{\beta-1} \quad \text{for } 0 < \theta < 1$$

$$X \,|\, \Theta = \theta \sim g(x|\theta)$$

likelihood

$$= \text{binomial}(n, x, \theta)$$

$$= \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$\therefore f(\theta \mid x) = \frac{h(\theta)\cdot g(x \mid \theta)}{\int_{\theta=0}^{\theta=1} h(\theta)\cdot g(x\mid\theta)\, d\theta}$$

$$= \frac{\cancel{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}\, \theta^{\alpha-1}(1-\theta)^{\beta-1} \cancel{\binom{n}{x}}\, \theta^{x}(1-\theta)^{n-x}}{\int_0^1 \cancel{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}\cdot \theta^{\alpha-1}(1-\theta)^{\beta-1}\cancel{\binom{n}{x}}\theta^{x}(1-\theta)^{n-x}\, dx}$$

$$= \frac{\theta^{\alpha+x-1}(1-\theta)^{(\beta+n-x)-1}}{\int_0^1 \theta^{(x+\alpha)-1}(1-\theta)^{(\beta+n-x)-1}\, d\theta}$$

= non-normalized beta pdf Beta($\alpha+x, \beta+n-x$)

$$= \frac{\theta^{(\alpha+x)-1}(1-\theta)^{(\beta+n-x)-1}}{\frac{\Gamma(\alpha+x)\cdot\Gamma(\beta+n-x)}{\Gamma(\alpha+\beta+n)}}$$

$$= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)\Gamma(\beta+n-x)}\cdot \theta^{(\alpha+x)-1}(1-\theta)^{(\beta+n-x)-1}$$

$$= \mathrm{Beta}(\alpha+x, \beta+n-x)$$

QED.

Other conjugacy relations:

- $h(\theta) \sim \gamma$ and $g(x|\theta) \sim \text{Poisson}(\theta)$.

  $\therefore f(\theta|x) = \gamma'$

- $h(\theta) \sim \text{Normal}$ and $g(x|\theta) \sim \text{Normal}'$

  $\therefore f(\theta|x) = \text{Normal}''$

  (normal is self-conjugate)

Hyper priors

$$\gamma \sim u(\gamma)$$

$$\theta \sim h(\theta|\gamma)$$

(solve w/ MCMC | Gibbs sampling)

Conjugate Priors

Prior h(Θ)	**Likelihood g(x | Θ)**	**Posterior f(Θ | x)**
$B(\alpha,\beta)$: $\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$	$Binomial$: $\dbinom{n}{x}p^{x}(1-p)^{n-x}$	$B(\alpha+x,\beta+n-x)$
$\Gamma(\alpha,\beta)$: $\dfrac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^{\alpha}}$ *if* $\theta\geq 0$	$Poisson$: $\dfrac{e^{-\theta}\theta^{x}}{x!}$	$\Gamma\left(\alpha+x,\dfrac{\beta}{n\beta+1}\right)$
$N(\mu,\tau^{2})$: $\dfrac{Exp\left(-\dfrac{(\theta-\mu)^{2}}{2\tau^{2}}\right)}{\tau\sqrt{2\pi}}$	$N(\theta,\sigma^{2})$: $\dfrac{Exp\left(-\dfrac{(x-\theta)^{2}}{2\sigma^{2}}\right)}{\sigma\sqrt{2\pi}}$	$N\left(\dfrac{\sigma^{2}\mu+\tau^{2}}{\sigma^{2}+\tau^{2}},\dfrac{\sigma^{2}\tau^{2}}{\sigma^{2}+\tau^{2}}\right)$